§5.

5.4. 构造全纯函数方法.

Lemma 5.5.

Let Ω be an open subset. Ω is symmetric

$\Omega^+ = \{z\in\Omega \mid z = x+iy,\ x,y\in\mathbb{R},\ y>0\}$.

Ω^-

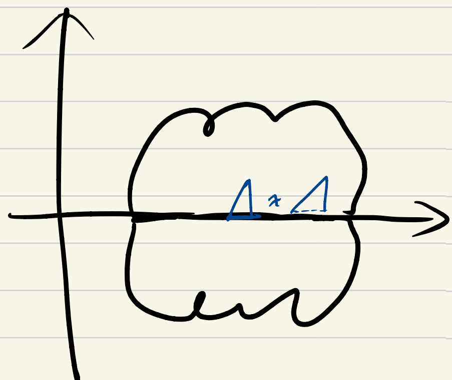
If $f^+ : \Omega^+ \to \mathbb{C}$ holomorphic function.

$f^- : \Omega^- \to \mathbb{C}$

f^+, f^- extend to I continuously.

$f^+(x) = f^-(x),\ \forall x\in I$. Then we define:

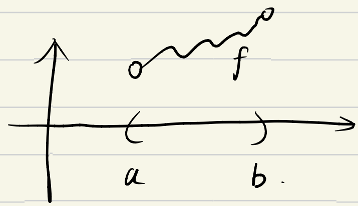
$$F(z) = \begin{cases} f^+(z), & z\in\Omega^+ \\ f^+(z) = f^-(z), & z\in I \\ f^-(z), & z\in\Omega^- \end{cases}$$



It suffices to show. $\forall T\cap I \neq \phi$, $\int_T F(z)dz = 0$.

1. 交一个顶点. ✓
2. I 截 T 两点. ✓
3. 一边相交. ✓ $\square$ Trivial.

f 唯一地延拓到 $[a,b]$.



Thm 5.6. $\Omega, \Omega^+, \Omega^-, I$. are defined as foregoing.

$f(z)$ 由 f in Ω^+ 确定.

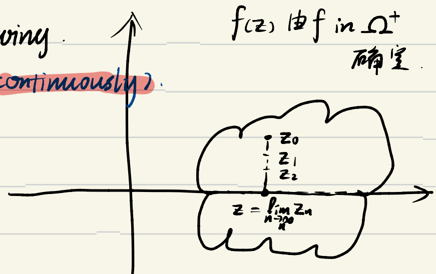
$f : \Omega^+ \to \mathbb{C}$, holomorphic. ($f$ can be extended to I continuously).

$f(x)\in\mathbb{R},\ \forall x\in I$.

then we can be extended to F as follow:

$$F(z) := \begin{cases} f(z), & z\in\Omega^+ \\ f(z), & z\in I \\ \overline{f(\bar z)}, & z\in\Omega^- \end{cases}$$

Then $F(z)$ is holomorphic in Ω.



$\overline{f(\bar x)} = f(x),\ \forall x\in I$.

It suffices to show, $\overline{f(\bar z)}$ is holomorphic. $I = \mathbb{R}\cap\Omega$.

要证: $f(x) = \overline{f(\bar x)}$

$z \in \Omega^- \Rightarrow \bar{z} \in \Omega^+ \Rightarrow f(\bar{z}) = \sum a_n (\bar{z} - \bar{z}_0)^n$, $\quad \bar{z}_0 \in \Omega^+$ $(z_0 \in \Omega^-)$.

$\Rightarrow F(z) = \sum \bar{a}_n (z - z_0)^n$. (in Ω^-).

$f^+ = F|_{\Omega^+}$. ✓

$f^- = F|_{\Omega^-}$. ✓ $\quad \square$.

· Remark: Schwarz $\Rightarrow$ Harmonic.

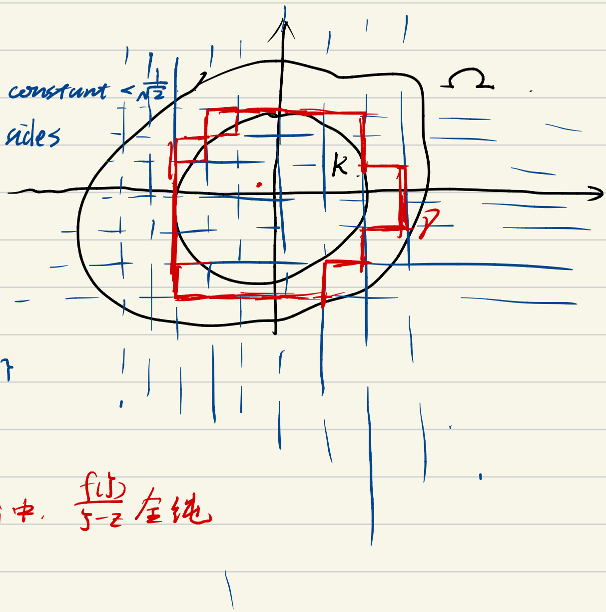
5.5 Runge's Approximation

Lemma 1: f holomorphic in an open set Ω.

$K \subseteq \Omega$ is compact.

则 ∃ 有限 $\gamma_1, \cdots, \gamma_N \subseteq \Omega - K$, such that:

$$f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta .$$



[Proof]: $d := c \cdot d(K, \Omega^c)$, where c is a constant $< \frac{1}{\sqrt{2}}$

consider a grid formed by solid squares with sides 平行于实、虚轴, 边长为 d.

Notice: $d(K, \Omega^c) > \sqrt{2} d$

若正方形与K有交, 则正方形 $\subseteq \Omega$.

即: 在 Ω 和 K 之间, 我们可以"画地为牢".

记 $\mathcal{Q} := \{Q_j \mid Q_j \cap K \neq \emptyset\} = \{Q_1, \cdots, Q_m\}$

$\forall z \in K$.

$z \in Q_{m_0}$ for some m_0.

$$2\pi i \int_{Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} 0. & 若\ j \neq m_0. \Rightarrow Q_j 中, \frac{f(\zeta)}{\zeta - z} 全纯 \\ f(z). & 若\ j = m_0. \end{cases}$$

$\Downarrow$

$$\sum_{j=1}^{m} 2\pi i \int_{Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z).$$

Lemma 2. f holomorphic in an open set Ω.

$K \subseteq \Omega$ is compact.

$\gamma \subseteq \Omega - K$ line segment. Then

∃ a sequence rational functions with singularities in γ that approximate f uniformly.

[Proof]: $\sum_{j=1}^{m} 2\pi i \int_{Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)$

1

suppose γ is parametrized by $\zeta : [0,1] \to \mathbb{C}$.

$$F(z,t) := 2\pi i \frac{f(\zeta(t))}{\zeta(t) - z} \quad : \quad K \times [0,1] \to \mathbb{C}$$

F is uniformly continuous

$\Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0$. s.t.

$\sup_{z \in K} |F(z,t_1) - F(z,t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta$.

$$f(z) = \int_\gamma 2\pi i \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 2\pi i \frac{f(\zeta(t))}{\zeta(t) - z} \zeta'(t)\, d\zeta$$

$$\Downarrow$$

$$\{f_n\} = \left\{ \underbrace{\sum_{n=1}^{N} 2\pi i \frac{f(\zeta(\frac{n}{N}))}{\zeta(\frac{n}{N}) - z} \zeta'(\tfrac{n}{N}) \cdot \frac{1}{N}}_{} \right\} 取极限.$$

is a rational funtion : $K \to \mathbb{C}$.

Hence rational functions $f_n \to f$ uniformly. $\square$.

Lemma 5.10. K^c is connected, $z_0 \in K^c$, then $\frac{1}{z - z_0}$ can be 逼近 by 多项式.

①. choose z_1. $z_1 \in D^c$, D a disc centered at O. $K \subseteq D$.

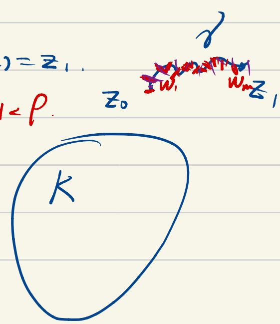
$$\frac{1}{z - z_1} = -\frac{1}{z_1} \cdot \frac{1}{1 - z/z_1} = \sum_{n=1}^{\infty} -\frac{z^n}{z_1^{n+1}} \quad (\in \mathbb{C}[[z]])$$

K^c is connected $\Rightarrow \exists \gamma : [0,1] \to \mathbb{C}$, $\gamma(0) = z_0$, $\gamma(1) = z_1$.

$\rho := \frac{1}{2} d(K, \gamma)$.

形式变换方法.

$|w_i - w_{i+1}| < \rho$.



$\forall w, w' \in \gamma$, $|w - w'| < \rho$

$$\frac{1}{z - w} = \frac{1}{z - w'} \cdot \frac{1}{1 - \frac{w - w'}{z - w'}} = \sum_{n \geq 0}^{\infty} \frac{(w - w')^n}{(z - w')^{n+1}}$$

$z - w' + w' - w$

" pull the singularities to infinity ".

f holomorphic in a neighborhood of a compact set K can be 逼近 uniformly by rational functions ✓

Further, if K^c is connected, then f 可被多项式逼近
polynomial

$\square$